The Joint Replenishment Problem
Optimal Policy And Exact Evaluation Method

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- General idea:
  - You keep several Stock Keeping Units (SKUs) in inventory.
  - For each SKU i, you incur a holding cost h_i and face a Poisson demand with rate parameter λ_i.
  - You can replenish the inventory of an SKU by issuing an order that has major order cost K. For each SKU i included in the order, you incur minor order cost k_i.

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  - You can replenish the inventory of an SKU by issuing an order that has major order cost K. For each SKU i included in the order, you incur minor order cost k_i.
- Million-dollar question: how do we coordinate orders such that holding and order costs are minimized?

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  - Can-order policy

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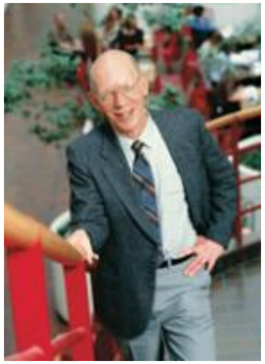
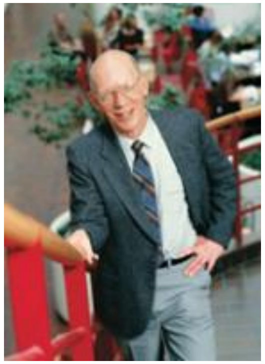
- The exact cost of a can-order policy can be determined using a Continuous-Time Markov Chain (CTMC)
- However, for systems with more than a few SKUs, the CTMC becomes too big, and we can no longer determine the best can-order policy (curse of dimensionality!)



Decomposition approach



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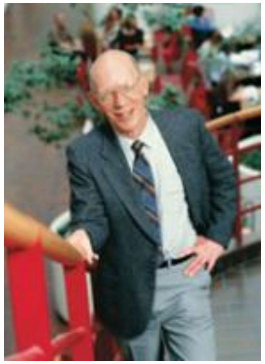

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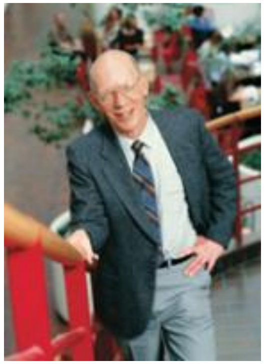

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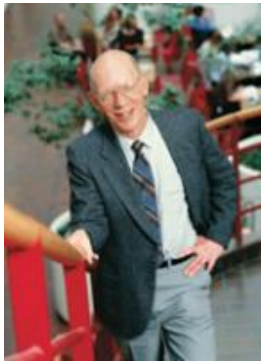

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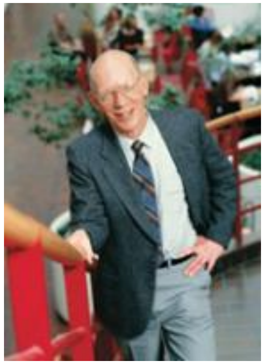

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  - Given the updated can-order policy for SKU i, determine the new rate of special replenishment opportunities μ_j for all other SKUs $j \neq i$.
  - Repeat this procedure for each SKU until the can-order policy itself convergences.

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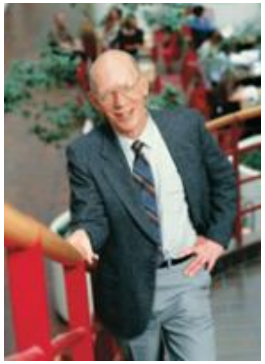

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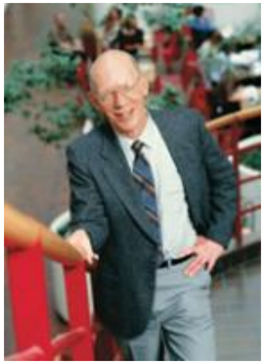

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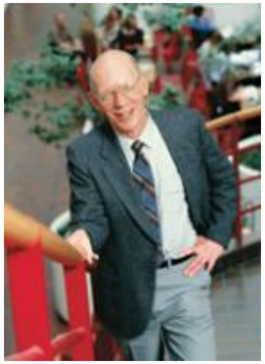

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  - It is a heuristic procedure (the single-item problem for SKU i ignores the interaction of SKUs $j \neq i$; all interaction is captured by a single parameter μ_i).
  - It approximates the cost of a single-item system using a closed-form expression. As a result, we need to simulate the can-order policy in order to obtain its real cost. In addition, to determine whether one can-order policy is better than another, we base ourselves on approximate costs (that may differ substantially from the real cost).

Main contributions

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- New, exact method to determine the cost of a JRP that partially solves the curse of dimensionality

New, exact method

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- We propose a new approach that uses a Discrete-Time Markov Chain (DTMC) that models transitions between so-called "initial states"; states in which we end up after an order has been triggered. By considering only initial states, we can reduce the number of states in our DTMC to $\sum_{i=1}^{N}\prod_{j\neq i}(S_i - c_i)$.

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- The reduction in the number of states can be significant:

Number of states required for analyzing the best can-order policy for the Federgruen instances			
Example problem	1	2	3
Traditional CTMC	34,848	34,848	18,000
New DTMC	256	300	853

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- In addition, we can easily extend our method (compound Poisson demand, lead time, backlog, lost sales…) without increasing the number of states.

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- Problem with two SKUs:
  - $\lambda_1 = 12$ and $\lambda_2 = 16$
  - $h_1 = 12$ and $h_2 = 23$
  - $k_1 = 7$ and $k_2 = 21$
  - $K = 25$
- No lead time
- Best can-order policy:
  - $S_1 = 7$, $c_1 = 4$, and $s_1 = 0$
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Can-order policy

(I_1, I_2)	→	(I_1, I_2)
(0,8)	→	(7,8)
(0,7)	→	(7,7)
(0,6)	→	(7,6)
(0,5)	→	(7,5)
(0,4)	→	(7,4)
(0,3)	→	(7,3)
(0,2)	→	(7,8)
(0,1)	→	(7,8)
(7,0)	→	(7,8)
(6,0)	→	(6,8)
(5,0)	→	(5,8)
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(3,0)	→	(7,8)
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- Example
- Two important implications:
  - The can-order policy is a logical heuristic; it adopts the structure of the optimal policy.
  - However, the can-order policy assumes a single can-order level for each SKU independent of the inventory levels of the SKUs that do not join the order ➔ if the number of SKUs increases, the optimality gap is expected to increase as well!

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- Using the insights of the optimal JRP policy, the can-order policy can be generalized using a greedy procedure:
  - Start from the best can-order policy.
  - For each combination of inventory levels of SKUs that do not join the order, evaluate whether it is beneficial to alter the can-order level (and/or order-up-to level) of SKUs that do join the order.
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- After applying this to the Federgruen instances, we get:

Expected cost of can-order policy and generalized can-order policy			
Example problem	1	2	3
Best can-order policy	77.51	80.87	67.80
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- In addition, rather than using an approximate (closed-form) cost function, we use our method to analyze the exact cost of the single/double/triple-item problems.

Expected cost of different policies for the Federgruen instances			
Example problem	1	2	3
Decomposition approach (approximation)	88.71	89.98	71.53
Decomposition approach (exact)	81.03	83.62	68.52
Generalized decomposition (single item)	80.07	82.66	68.70
Generalized decomposition (double item)	78.10	82.16	68.04
Generalized decomposition (triple item)	77.97	81.27	67.96
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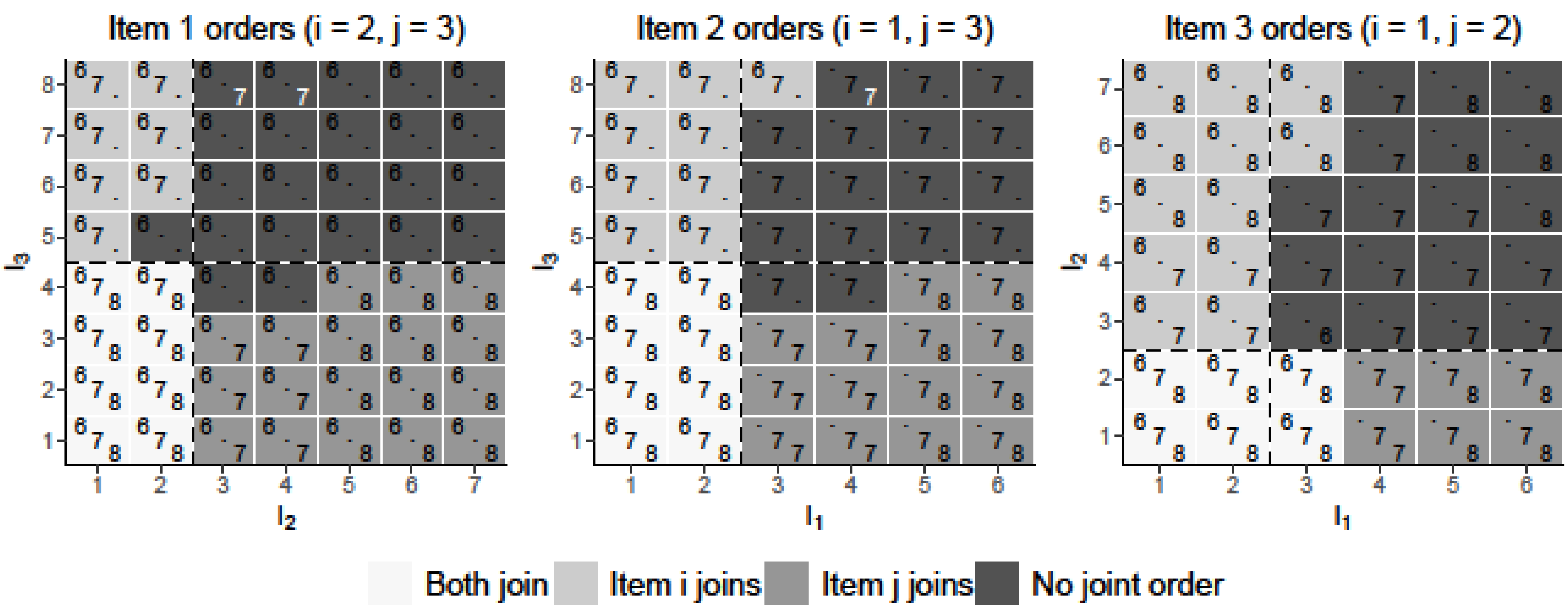

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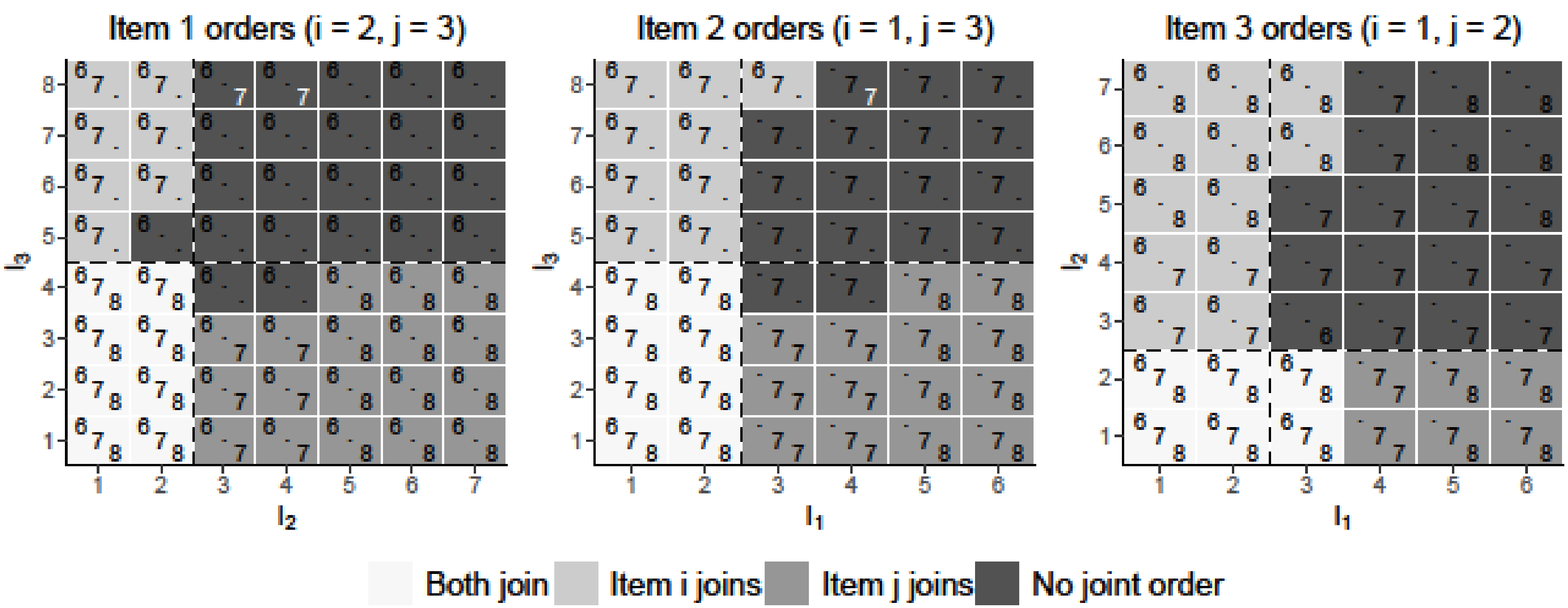
  - $S_1 = 6$, $c_1 = 2$, and $s_1 = 0$
  - $S_2 = 7$, $c_2 = 2$, and $s_2 = 0$
  - $S_3 = 7$, $c_3 = 4$, and $s_3 = 0$
- It may be optimal to return inventory!



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